

Convergent perturbative power series solution of the stationary Maxwell–Born–Infeld field equations with regular sources

M.K.-H. KIESSLING

*Department of Mathematics,
Rutgers, The State University of New Jersey,
110 Frelinghuysen Rd., Piscataway, NJ 08854*

Abstract

The stationary Maxwell-Born-Infeld field equations of electromagnetism with regular sources $\rho \in (C_0^\alpha \cap L^1)(\mathbb{R}^3)$ and $j \in (C_0^\alpha \cap L^1)(\mathbb{R}^3)$ (componentwise) are solved using a perturbation series expansion in powers of Born's electromagnetic constant. The convergence in $C_0^{1,\alpha}$ of the power series for the fields is proved with the help of Banach algebra arguments and complex analysis. The finite radius of convergence depends on the “ $C_0^{1,\alpha}$ size” of both, the Coulomb field generated by ρ and the Ampère field generated by j . No symmetry is assumed.

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1 Introduction

In a previous paper [CaKi2010], the author and Holly Carley developed a constructive approach to the prescribed mean-curvature equation [GiTr1983]

$$\pm \nabla \cdot \frac{\nabla u}{\sqrt{1 \pm |\nabla u|^2}} = nH \quad (1)$$

for hypersurfaces which are graphs of some scalar function u over \mathbb{R}^n , having mean-curvature $H \in (C_0^\alpha \cap L^1)(\mathbb{R}^3)$ suitably small. In (1) the $+$ sign refers to hypersurfaces in Euclidean \mathbb{R}^{n+1} , while the $-$ sign refers to spacelike hypersurfaces in Minkowski spacetime $\mathbb{R}^{1,n}$. This latter setting, when $n = 3$, is equivalent to the electrostatic Maxwell–Born–Infeld problem for the electric potential $\phi \propto u$ with regular charge density $\rho \propto H$ and small Born constant β . In the present paper this constructive approach is generalized to the stationary Maxwell–Born–Infeld field equations for the electric and magnetic potentials ϕ and A , or rather their electric and magnetic fields E and B , with regular charge density ρ and current vector-density j . We here are interested only in electromagnetic fields in unbounded space \mathbb{R}^3 with vanishing conditions at spatial infinity.

Since the Maxwell–Born–Infeld field equations were proposed as a nonlinear remedy for the infinite field energies and momenta incurred with point charge sources in the linear Maxwell–Maxwell field equations (the Maxwell–Lorentz theory), our study of the Maxwell–Born–Infeld field equations with regular source terms ρ and j may seem irrelevant from a fundamental physical perspective. However, sufficiently far away from delta function sources the fields should be practically indistinguishable from regularizations thereof, yet very little is known about solving the Maxwell–Born–Infeld field equations for non-special choices of source terms.

In the next section we briefly summarize the stationary Maxwell–Maxwell and Maxwell–Born–Infeld field equations in dimensionless units in which the speed of light $c = 1$. There we also explain our terminology. Section 3 puts forward our perturbative strategy to solve these equations. In section 4 we prove the convergence of the perturbative, formal power series solution. In section 5 we specialize to the electrostatic and magnetostatic subproblems. Section 6 concludes the main part of this paper with a brief summary and an outlook on applications. The appendix summarizes some important details concerning the recurrence relation for the coefficients of the perturbative power series which were supplied by my colleague Doron Zeilberger in collaboration with Shalosh B. Ekhad III.

2 Stationary electromagnetic Maxwell–Born–Infeld theory

Maxwell’s classical electromagnetic theory comprises Coulomb’s law, Faraday’s law, the Ampère–Maxwell law, and the unnamed law of the absence of magnetic charges. In stationary situations the Ampère–Maxwell law reduces to Ampère’s law, and Faraday’s law to its stationary special case. Coulomb’s law states that an electrostatic charge density ρ in \mathbb{R}^3 is the source for the electric displacement field D ,

$$\nabla \cdot D = 4\pi\rho \quad \text{Coulomb's law,} \quad (2)$$

while Faraday’s law says that the electric field E , when stationary, is curl-free:

$$\nabla \times E = 0 \quad \text{Faraday's law (stationary).} \quad (3)$$

Similarly, Ampère’s law states that a solenoidal electric current vector-density j in \mathbb{R}^3 is the source for the magnetic excitation field H ,

$$\nabla \times H = 4\pi j \quad \text{Ampère's law,} \quad (4)$$

while the law of the absence of sources of the magnetic induction field B reads

$$\nabla \cdot B = 0 \quad \text{no – magnetic – charges law.} \quad (5)$$

The four fields $E, D; B, H$ need to be linked by an “aether law,” with the help of which one can eliminate one of the electric-, and one of the magnetic-type fields from the equations and obtain a closed, nonlinear set of first-order pde for the two remaining fields. Note that for us “aether” is just convenient shorthand for “electromagnetic vacuum;” of course, for Maxwell and his contemporaries it had a substantial meaning until it was demolished by Einstein. Maxwell’s law of the “pure aether” reads

$$E = D \quad \text{Maxwell's E – law;} \quad (6)$$

$$B = H \quad \text{Maxwell's B – law;} \quad (7)$$

whereas the aether law of Born and Infeld [BoIn1933/34] says

$$E = \frac{D - \beta^4 D \cdot H H}{\sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2}} \quad \text{Born – Infeld's E – law;} \quad (8)$$

$$B = \frac{H + \beta^4 H \cdot D D}{\sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2}} \quad \text{Born – Infeld's B – law.} \quad (9)$$

In (8) and (9), $\beta \in (0, \infty)$ is a hypothetical new constant of nature (in the dimensionless notation of [Kie2004a]), which we have called Born's aether constant. In the limit $\beta \rightarrow 0$ the Born–Infeld law goes over into Maxwell's aether law.

To avoid any ambiguity as to which set of field equations we refer to in this paper (and elsewhere) we will always use the combination “Maxwell–whose aether law.” Thus, in particular, we will speak of the Maxwell–Maxwell equations when normally one just refers to the Maxwell equations.

3 Formal perturbation theory

Assume that the sources $\rho \in (C_0^\alpha \cap L^1)(\mathbb{R}^3)$, and $j \in (C_0^\alpha \cap L^1)(\mathbb{R}^3)$ componentwise. We seek stationary solutions in unbounded space \mathbb{R}^3 of the electromagnetic Maxwell–Born–Infeld field equations satisfying vanishing conditions at spatial infinity.

We begin by recalling the just mentioned fact that for $\beta \rightarrow 0$ the system of Maxwell–Born–Infeld field equations reduces to the system of Maxwell–Maxwell field equations, which for the stipulated class of sources and asymptotic conditions are uniquely solved by $E_C = D_C$ and $B_A = H_A$, where

$$D_C(x) := -\nabla \int \frac{\rho(y)}{|x-y|} d^3y \quad (10)$$

$$H_A(x) := \nabla \times \int \frac{j(y)}{|x-y|} d^3y. \quad (11)$$

Here, the subscripts $_C$ and $_A$ stand for Coulomb and Ampère respectively. This suggests that for $\beta > 0$ we may seek a solution to the Maxwell–Born–Infeld field equations for the same sources and asymptotic conditions as for the Maxwell–Maxwell field equations by setting $D_{BI} = D_C + D_\beta$ and $H_{BI} = H_A + H_\beta$, with $\lim_{\beta \rightarrow 0} D_\beta = 0$ and $\lim_{\beta \rightarrow 0} H_\beta = 0$, from which the electric field E_{BI} and magnetic induction field B_{BI} are obtained through the Born–Infeld aether law.

To determine D_β and H_β we first note that, since $\nabla \cdot D_C = 4\pi\rho$, we have that

$$\nabla \cdot D_\beta = 0, \quad (12)$$

and since $\nabla \times H_A = 4\pi j$, we have that

$$\nabla \times H_\beta = 0. \quad (13)$$

Thus, D_β is a curl field, while H_β is a gradient field.

Incidentally, since moreover D_C is a gradient field and H_A a curl field, we now see that the Ansatz $D_{BI} = D_C + D_\beta$ and $H_{BI} = H_A + H_\beta$ corresponds precisely to the Helmholtz decomposition for the fields D_{BI} and H_{BI} . For each field we have one of the two Helmholtz components and seek the remaining one.

Coming back to the determination of D_β and H_β , we now need a curl equation for D_β and a divergence equation for H_β . So, using (8) to eliminate E in favor of D and H in (3), and using (9) to eliminate B in favor of D and H in (5), we obtain a system of nonlinear vector pde for D_β and H_β which is closed *conditioned on D_C and H_A being given*. This systems of conditional vector pde reads

$$\nabla \times \frac{D - \beta^4 D \cdot H H}{\sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2}} = 0, \quad (14)$$

$$\nabla \cdot \frac{H + \beta^4 H \cdot D D}{\sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2}} = 0, \quad (15)$$

where D and H stand for $D_C + D_\beta$ and H for $H_A + H_\beta$, respectively.

The analyticity properties of the nonlinearities now suggest in particular that a solution pair D_β, H_β for the nonlinear first-order vector problem for small but finite β is itself analytic in β , i.e. satisfies a power series Ansatz, more specifically

$$D_\beta(x) = \sum_{k=1}^{\infty} \beta^{4k} D^{(k)}(x), \quad (16)$$

$$H_\beta(x) = \sum_{k=1}^{\infty} \beta^{4k} H^{(k)}(x), \quad (17)$$

with each $D^{(k)}$ and $H^{(k)}$ independent of β . Inserting this power series Ansatz into the pair of equations (14), (15) and sorting according to powers of β , we find a hierarchy of linear equations, the k -th members of which read:

$$\nabla \times D^{(k)} = \nabla \times \sum_{h=1}^k \left(X^{(h)} D^{(k-h)} + Y^{(h)} H^{(k-h)} \right); \quad k \in \mathbb{N} \quad (18)$$

(together with $\nabla \cdot D^{(k)} = 0$), and

$$\nabla \cdot H^{(k)} = \nabla \cdot \sum_{h=1}^k \left(X^{(h)} H^{(k-h)} - Y^{(h)} D^{(k-h)} \right); \quad k \in \mathbb{N} \quad (19)$$

(together with $\nabla \times H^{(k)} = 0$); here, $D^{(0)} := D_C$ and $H^{(0)} := H_A$, and $X^{(h)}$ and $Y^{(h)}$

are the coefficients of β^{4h} in the power series expansions of the two formal functions

$$\beta^4 \mapsto -1/\sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2}$$

and

$$\beta^4 \mapsto \beta^4 D \cdot H / \sqrt{1 + \beta^4(|D|^2 - |H|^2) - \beta^8(H \cdot D)^2},$$

respectively, with $D = D_C + D_\beta$ and $H = H_A + H_\beta$. Explicitly, and including $h = 0$, we have $X^{(0)} = -1$, while for $h \in \mathbb{N}$ we have

$$X^{(h)} = - \sum_{j=1}^h M_j \sum_{l=0}^j (-1)^{j-l} \binom{j}{l} Z_{h,j,l}, \quad (20)$$

with

$$Z_{h,j,0} = \sum_{|b|_{4j}=h-2j} \prod_{n=1}^{2j} D^{(b_{2n-1})} \cdot H^{(b_{2n})}, \quad (21)$$

$$Z_{h,j,j} = \sum_{|a|_{2j}=h-j} \prod_{m=1}^j (D - H)^{(a_{2m-1})} \cdot (D + H)^{(a_{2m})}, \quad (22)$$

and, for $0 < l < j$,

$$Z_{h,j,l} = \sum_{\substack{|a|_{2l} + |b|_{4(j-l)} \\ = h + l - 2j}} \prod_{m=1}^l (D - H)^{(a_{2m-1})} \cdot (D + H)^{(a_{2m})} \prod_{n=1}^{2(j-l)} D^{(b_{2n-1})} \cdot H^{(b_{2n})}. \quad (23)$$

Here, $|\ell|_K := \sum_{i=1}^K \ell_i$, with ℓ_i taking any non-negative integer values; so in particular, $\sum_{|\ell|_K < 0} \dots = 0$. Also, $M_0 := 1$ and

$$M_j = (-1)^j \frac{(2j-1)!!}{j! 2^j}; \quad j \in \mathbb{N} \quad (24)$$

are the Maclaurin coefficients for $1/\sqrt{1+z}$. Having $X^{(h)}$ for $h = 0, 1, 2, \dots$, we find

$$Y^{(h)} = - \sum_{g=0}^{h-1} X^{(h-1-g)} \sum_{|a|_2=g} D^{(a_1)} \cdot H^{(a_2)}; \quad h \in \mathbb{N}. \quad (25)$$

Note that $X^{(h)}$ and $Y^{(h)}$ contain only terms of order $< h$ of $|D_\beta|^2$, $|H_\beta|^2$, and $D_\beta \cdot H_\beta$.

The expressions for $X^{(k)}$ and $Y^{(k)}$ look somewhat unwieldy, but since there is scant empirical evidence (if any!) for the need of corrections to the solutions of the Maxwell–Maxwell field equations in the regime of classical phenomena, for many practical applications it may suffice to just compute the first correction term to the Coulomb and Ampère fields. These are of a more manageable structure. In particular, $D^{(1)}$ satisfies

$$\nabla \times D^{(1)} = \nabla \times \left(\frac{1}{2}(|D_C|^2 - |H_A|^2)D_C + D_C \cdot H_A H_A \right), \quad (26)$$

together with $\nabla \cdot D^{(1)} = 0$, and $H^{(1)}$ satisfies

$$\nabla \cdot H^{(1)} = \nabla \cdot \left(\frac{1}{2}(|D_C|^2 - |H_A|^2)H_A - D_C \cdot H_A D_C \right), \quad (27)$$

together with $\nabla \times H^{(1)} = 0$.

As to the solvability of the linear first-order PDE (18), (19), assuming that their right-hand sides are in C_0^α , this pair of PDE has a unique solution in $C_0^{1,\alpha}$ given by

$$D^{(k)} = \mathbf{P} \sum_{h=1}^k \left(X^{(h)} D^{(k-h)} + Y^{(h)} H^{(k-h)} \right), \quad k \in \mathbb{N} \quad (28)$$

$$H^{(k)} = \mathbf{Q} \sum_{h=1}^k \left(X^{(h)} H^{(k-h)} - Y^{(h)} D^{(k-h)} \right), \quad k \in \mathbb{N} \quad (29)$$

where $\mathbf{P} : C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ projects onto the solenoidal subspace of $C_0^{1,\alpha}$, and where $\mathbf{Q} : C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ projects onto the curl-free subspace of $C_0^{1,\alpha}$. More explicitly, for a vector field $V^{(k)} \in C_0^{1,\alpha}$ with $\nabla \cdot V^{(k)} \in C_0^{0,\alpha} \cap L^1$, we have

$$\mathbf{P}V^{(k)}(x) = V^{(k)}(x) + \nabla \int \frac{\frac{1}{4\pi} \nabla \cdot V^{(k)}(y)}{|x-y|} d^3y, \quad (30)$$

and when $\nabla \times V^{(k)} \in C_0^{0,\alpha} \cap L^1$, we have

$$\mathbf{Q}V^{(k)}(x) = V^{(k)}(x) - \nabla \times \int \frac{\frac{1}{4\pi} \nabla \times V^{(k)}(y)}{|x-y|} d^3y. \quad (31)$$

To summarize, so far we have seen that the Ansatz $D = D_C + \sum_{k=1}^\infty \beta^{4k} D^{(k)}$ and $H = H_A + \sum_{k=1}^\infty \beta^{4k} H^{(k)}$, with $D^{(k)}$ and $H^{(k)}$ for $k \in \mathbb{N}$ recursively given by (28) and (29), yields a *formal* series solution pair for the stationary Maxwell–Born–Infeld field equations with regular sources. We next prove its convergence in $C_0^{1,\alpha}$ for small enough β , given the sources $\rho \in C_0^\alpha$ and $j \in C_0^\alpha$.

4 Rigorous perturbation theory modulo some MAPLE calculations

4.1 $C_0^{1,\alpha}$ consistency of the formal perturbation series solution

We first confirm that the formal series $D_\beta = \sum_{k=1}^{\infty} \beta^{4k} D^{(k)}$ and $H_\beta = \sum_{k=1}^{\infty} \beta^{4k} H^{(k)}$ are $C_0^{1,\alpha}$ series. For this we have to check that all $D^{(k)}$ and $H^{(k)}$ as given in (30) and (31) are in $C_0^{1,\alpha}$. But this follows inductively from the facts that $D^{(0)} = D_C \in C_0^{1,\alpha}$ and $H^{(0)} = H_A \in C_0^{1,\alpha}$, that $C_0^{1,\alpha}$ is a Banach algebra, and that $\mathbf{P} : C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ and $\mathbf{Q} : C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$; hence, each partial sum of $D_\beta = \sum_{k=1}^{\infty} \beta^{4k} D^{(k)}$ is in $C_0^{1,\alpha}$ and so is each partial sum of $H_\beta = \sum_{k=1}^{\infty} \beta^{4k} H^{(k)}$.

4.2 Absolute convergence of the formal perturbation series solution

Since $\|D_\beta\| \leq \sum_{k=1}^{\infty} \beta^{4k} \|D^{(k)}\|$ and $\|H_\beta\| \leq \sum_{k=1}^{\infty} \beta^{4k} \|H^{(k)}\|$ in the sense of partial sums, to prove absolute convergence of the power series for D_β and H_β it suffices to show that $\sum_{k=1}^{\infty} \beta^{4k} (\|D^{(k)}\| + \|H^{(k)}\|) < \infty$ for sufficiently small β , given ρ and j .

We first estimate $\|D^{(k)}\| + \|H^{(k)}\|$ for $k \in \mathbb{N}$ in terms of the $\|D^{(l)}\| + \|H^{(l)}\|$ for all $l < k$. Beginning with $\|D^{(k)}\|$, for $k \in \mathbb{N}$ we find

$$\|D^{(k)}\| = \|\mathbf{P} \sum_{h=1}^k (X^{(h)} D^{(k-h)} + Y^{(h)} H^{(k-h)})\| \quad (32)$$

$$\leq \left\| \sum_{h=1}^k (X^{(h)} D^{(k-h)} + Y^{(h)} H^{(k-h)}) \right\| \quad (33)$$

$$\leq \sum_{h=1}^k \left\| (X^{(h)} D^{(k-h)} + Y^{(h)} H^{(k-h)}) \right\| \quad (34)$$

$$\leq \sum_{h=1}^k (\|X^{(h)} D^{(k-h)}\| + \|Y^{(h)} H^{(k-h)}\|) \quad (35)$$

$$\leq \sum_{h=1}^k (\|X^{(h)}\| \|D^{(k-h)}\| + \|Y^{(h)}\| \|H^{(k-h)}\|). \quad (36)$$

Here, inequality (33) holds because $\mathbf{P} : C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ is a projector, inequalities (34) and (35) are just the triangle inequality, while (36) is valid in Banach algebras (here $C_0^{1,\alpha}$). Similarly, for $\|H^{(k)}\|$ and all $k \in \mathbb{N}$ we find

$$\|H^{(k)}\| \leq \sum_{h=1}^k (\|X^{(h)}\| \|H^{(k-h)}\| + \|Y^{(h)}\| \|D^{(k-h)}\|). \quad (37)$$

Adding these estimates for $\|D^{(k)}\|$ and $\|H^{(k)}\|$ yields

$$\|D^{(k)}\| + \|H^{(k)}\| \leq \sum_{h=1}^k (\|X^{(h)}\| + \|Y^{(h)}\|) (\|D^{(k-h)}\| + \|H^{(k-h)}\|). \quad (38)$$

Since $\|X^{(h)}\|$ and $\|Y^{(h)}\|$ depend in turn on $\|D^{(l)}\|$ and $\|H^{(l)}\|$, we need to carry on and estimate $\|X^{(h)}\| + \|Y^{(h)}\|$ in terms of the $\|D^{(l)}\| + \|H^{(l)}\|$ for all $l < k$. As to $\|X^{(h)}\|$, we have $\|X^{(0)}\| = 1$ and

$$\|X^{(h)}\| \leq \sum_{j=1}^h |M_j| \sum_{l=0}^j \binom{j}{l} \|Z_{h,j,l}\| \quad (39)$$

and, for $h \in \mathbb{N}$,

$$\|Z_{h,j,0}\| \leq \sum_{|b|_{4j}=h-2j} \prod_{n=1}^{2j} \|D^{(b_{2n-1})}\| \|H^{(b_{2n})}\| \quad (40)$$

$$\leq \sum_{|b|_{4j}=h-2j} \prod_{n=1}^{4j} \left(\|D^{(b_n)}\| + \|H^{(b_n)}\| \right), \quad (41)$$

$$\|Z_{h,j,j}\| \leq \sum_{|a|_{2j}=h-j} \prod_{m=1}^j \|(D-H)^{(a_{2m-1})}\| \|(D+H)^{(a_{2m})}\| \quad (42)$$

$$\leq \sum_{|a|_{2j}=h-j} \prod_{m=1}^{2j} \left(\|D^{(a_m)}\| + \|H^{(a_m)}\| \right), \quad (43)$$

and if $j > 1$, which can happen only when $h \geq 2$, then for $0 < l < j$ we have

$$\|Z_{h,j,l}\| \leq \sum_{\substack{|a|_{2l}+|b|_{4(j-l)} \\ =h+l-2j}} \prod_{m=1}^l \|(D-H)^{(a_{2m-1})}\| \|(D+H)^{(a_{2m})}\| \prod_{n=1}^{2(j-l)} \|D^{(b_{2n-1})}\| \|H^{(b_{2n})}\| \quad (44)$$

$$\leq \sum_{\substack{|a|_{2l}+|b|_{4(j-l)} \\ =h+l-2j}} \prod_{m=1}^{2l} \left(\|D^{(a_m)}\| + \|H^{(a_m)}\| \right) \prod_{n=1}^{4(j-l)} \left(\|D^{(b_n)}\| + \|H^{(b_n)}\| \right). \quad (45)$$

Finally, as to $\|Y^{(h)}\|$, we have

$$\|Y^{(h)}\| \leq \sum_{g=0}^{h-1} \|X^{(h-1-g)}\| \sum_{|a|_2=g} \|D^{(a_1)}\| \|H^{(a_2)}\| \quad (46)$$

$$\leq \sum_{g=0}^{h-1} \|X^{(h-1-g)}\| \sum_{|a|_2=g} \left(\|D^{(a_1)}\| + \|H^{(a_1)}\| \right) \left(\|D^{(a_2)}\| + \|H^{(a_2)}\| \right). \quad (47)$$

Having completed our estimate of $\|D^{(k)}\| + \|H^{(k)}\|$ for $k \in \mathbb{N}$ in terms of the $\|D^{(l)}\| + \|H^{(l)}\|$ for all $l < k$, we now abbreviate $\|D^{(n)}\| + \|H^{(n)}\| = N^{(n)}$ and recast our estimates more user-friendly as

$$N^{(k)} \leq \sum_{h=1}^k \left(\|X^{(h)}\| + \sum_{g=0}^{h-1} \|X^{(h-1-g)}\| \sum_{|a|_2=g} N^{(a_1)} N^{(a_2)} \right) N^{(k-h)}, \quad (48)$$

together with $\|X^{(0)}\| = 1$, then

$$\|X^{(1)}\| \leq \frac{1}{2} \sum_{|a|_2=0}^2 \prod_{m=1}^2 N^{(a_m)} = \frac{1}{2} N^{(0)^2}, \quad (49)$$

and, for $h \geq 2$,

$$\begin{aligned} \|X^{(h)}\| \leq & |M_1| \left[\sum_{|b|_4=h-2} \prod_{n=1}^4 N^{(b_n)} + \sum_{|a|_2=h-1} \prod_{m=1}^2 N^{(a_m)} \right] \\ & + \sum_{j=2}^h |M_j| \left[\sum_{|b|_{4j}=h-2j} \prod_{n=1}^{4j} N^{(b_n)} + \sum_{|a|_{2j}=h-j} \prod_{m=1}^{2j} N^{(a_m)} \right. \\ & \left. + \sum_{l=1}^{j-1} \binom{j}{l} \sum_{\substack{|a|_{2l}+|b|_{4(j-l)} \\ =h+l-2j}} \prod_{m=1}^{2l} N^{(a_m)} \prod_{n=1}^{4(j-l)} N^{(b_n)} \right]. \end{aligned} \quad (50)$$

By an inductive argument we now estimate each $N^{(k)}$ in terms of the $2k+1$ -th power of $N^{(0)} =: \mathcal{N}$.

In this vein, setting $k = 1$ in (48) we obtain the estimate

$$N^{(1)} \leq \left(\|X^{(1)}\| + \mathcal{N}^2 \right) \mathcal{N}, \quad (51)$$

and for $\|X^{(1)}\|$ we have (49), so

$$N^{(1)} \leq \frac{3}{2} \mathcal{N}^3. \quad (52)$$

Next, suppose that for all $k = 1, \dots, k_*$ there exists some R_k such that

$$N^{(k)} \leq R_k \mathcal{N}^{2k+1}. \quad (53)$$

Also set $R_0 := 1$ (for, $N^{(0)} = \mathcal{N}$). Inserting these estimates into (48) then yields

$$N^{(k_*+1)} \leq \sum_{h=1}^{k_*+1} \left[\|X^{(h)}\| + \sum_{g=0}^{h-1} \|X^{(h-1-g)}\| \mathcal{N}^{2g+2} \sum_{|a|_2=g} R_{a_1} R_{a_2} \right] R_{k_*+1-h} \mathcal{N}^{2(k_*+1-h)+1}. \quad (54)$$

As to the $\|X^{(h)}\|$, for $h = 1$ we already have the estimate (49), and inserting (53) into (50) we find for $h \geq 2$ that

$$\|X^{(h)}\| \leq |M_1| \left[\sum_{|b|_4=h-2} \prod_{n=1}^4 R_{b_n} \mathcal{N}^{2b_n+1} + \sum_{|a|_2=h-1} \prod_{m=1}^2 R_{a_m} \mathcal{N}^{2a_m+1} \right] \quad (55)$$

$$+ \sum_{j=2}^h |M_j| \left[\sum_{|b|_{4j}=h-2j} \prod_{n=1}^{4j} R_{b_n} \mathcal{N}^{2b_n+1} + \sum_{|a|_{2j}=h-j} \prod_{m=1}^{2j} R_{a_m} \mathcal{N}^{2a_m+1} \right. \\ \left. + \sum_{l=1}^{j-1} \binom{j}{l} \sum_{\substack{|a|_{2l}+|b|_{4(j-l)} \\ = h+l-2j}} \prod_{m=1}^{2l} R_{a_m} \mathcal{N}^{2a_m+1} \prod_{n=1}^{4(j-l)} R_{b_n} \mathcal{N}^{2b_n+1} \right]$$

$$= \mathcal{N}^{2h} \left(|M_1| \left[\sum_{|b|_4=h-2} \prod_{n=1}^4 R_{b_n} + \sum_{|a|_2=h-1} \prod_{m=1}^2 R_{a_m} \right] \right. \\ \left. + \sum_{j=2}^h |M_j| \left[\sum_{|b|_{4j}=h-2j} \prod_{n=1}^{4j} R_{b_n} + \sum_{|a|_{2j}=h-j} \prod_{m=1}^{2j} R_{a_m} \right. \right. \\ \left. \left. + \sum_{l=1}^{j-1} \binom{j}{l} \sum_{\substack{|a|_{2l}+|b|_{4(j-l)} \\ = h+l-2j}} \prod_{m=1}^{2l} R_{a_m} \prod_{n=1}^{4(j-l)} R_{b_n} \right] \right). \quad (56)$$

Estimates (49) and (55), (56) state that $\forall h \in \mathbb{N} : \exists C_h > 0$ such that $\|X^{(h)}\| \leq C_h \mathcal{N}^{2h}$. This also holds for $h = 0$, with $C_0 = 1$. Inserted back into (54) this shows now that (53) is true also for $k = k_* + 1$, and since $k_* \geq 1$ is arbitrary in this induction step while (52) says that (53) is true for $k_* = 1$, it follows that (53) is true for all $k \in \mathbb{N}$.

Our inductive argument also yields R_k as recursively defined for $k \in \mathbb{N}$ by

$$R_k = \sum_{h=1}^k \left(C_h + \sum_{g=0}^{h-1} C_{h-1-g} \sum_{|a|_2=g} R_{a_1} R_{a_2} \right) R_{k-h}, \quad (57)$$

with $R_0 := 1 =: C_0$, $C_1 = 1/2$ (49), and C_h for $h \geq 2$ given by $\text{coeff}(\mathcal{N}^{2h})$ at rhs(56).

To summarize, so far we have proved that the formal power series solutions for D_β and H_β satisfy

$$\sum_{k=1}^{\infty} \beta^{4k} (\|D^{(k)}\| + \|H^{(k)}\|) \leq \sum_{k=1}^{\infty} R_k \beta^{4k} \mathcal{N}^{2k+1} \quad (58)$$

with the R_k given as stated in (57) and its ensuing text. We next discuss the convergence of rhs(58).

We begin by noting that the convergence of the series at rhs(58) is not altered if we multiply it by β^2 and add a term $\beta^2 \mathcal{N}$ to it. Now defining $\xi := \beta^2 \mathcal{N}$, as well as $c_1 := 1$ and $c_{2k+1} := R_k$ for $k \in \mathbb{N}$, we see that the convergence of rhs(58) is decided by the convergence of the series

$$G(\xi) := \sum_{k=0}^{\infty} c_{2k+1} \xi^{2k+1} \quad (59)$$

for $\xi \in \mathbb{C}$. Note that the formal power series (59) is just the generating function of the c_{2k+1} , i.e. $c_{2k+1} = G^{(2k+1)}(0)/(2k+1)!$, formally. We need to show that the generating function is analytic about $\xi = 0$ with non-zero radius of convergence.

With the help of the recursion relation (57) we readily find that for positive ξ the function $G(\xi)$ is the positive inverse of the locally analytic function $g \mapsto \xi$ given by

$$\xi = 2g - \frac{g + g^3}{\sqrt{1 - g^2 - g^4}}, \quad (60)$$

which has unit derivative at $g = 0$, and so its inverse $\xi \mapsto g$ is analytic in an open ξ -neighborhood of $\xi = 0$, with $G(0) = 0$. This argument already implies a finite radius of convergence, but without any information on its size.

Before we actually determine the radius of convergence we note that an upper bound for it is readily found by discussing the function $g \mapsto \xi$ defined in (60) on its real interval of definition about zero. Clearly, this interval is $(-g_\bullet, g_\bullet)$, where $g_\bullet = \sqrt{(\sqrt{5} - 1)/2} \approx 0.7861513775$ is the positive real zero of the square root term in (60). Furthermore, $g \mapsto \xi$ is odd, concave for positive and convex for negative g values, having a unique maximum at about $g^* = 0.4039458281$ and a unique minimum at $-g^*$, yielding the ξ -values $\xi^* = 0.285891853$ for the maximum, and $-\xi^*$ for the minimum; the numerical values are easily obtained using MAPLE. Clearly, ξ^* is an upper bound to the radius of convergence of the power series for $G(\xi)$.

We now show that ξ^* is the radius of convergence. In fact, the radius of convergence is found amongst those ξ values closest to $\xi = 0$ at which the derivative of $g \mapsto \xi$ vanishes (possibly asymptotically should $\xi \rightarrow \xi_\infty$ when $g \rightarrow \infty$ suitably). Now, since any solution $g = G(\xi)$ to (60) with $G(0) = 0$ is also a solution to the algebraic problem of finding the roots of the polynomial of degree 6 in g with ξ -dependent coefficients, given by

$$P(g|\xi) = (\xi - 2g)^2(1 - g^2 - g^4) - (g + g^3)^2, \quad (61)$$

the asymptotic scenario does not occur and all one needs to do is to locate the finitely many zeros of the g -derivative of $g \mapsto \xi$, which reads

$$\frac{d\xi}{dg} = -\frac{1 + 3g^2 - g^4 - g^6 - 2\sqrt{1 - g^2 - g^4}^3}{\sqrt{1 - g^2 - g^4}^3}. \quad (62)$$

Clearly, a zero of r.h.s.(62) is also a zero of the order-12 polynomial $\tilde{P}(g)$ obtained by multiplying the numerator of r.h.s.(62) by $1 + 3g^2 - g^4 - g^6 + 2\sqrt{1 - g^2 - g^4}^3$, viz.

$$\tilde{P}(g) = (1 + 3g^2 - g^4 - g^6)^2 - 4(1 - g^2 - g^4)^3. \quad (63)$$

By the fundamental theorem of algebra, $\tilde{P}(g)$ has twelve zeros, counted in multiplicity; six of these are zeros of $d\xi/dg$ given in (62). With the help of MAPLE one finds that if g_0 is a zero of $d\xi/dg$ given in (62), then

$$g_0 \in \pm\{0.4039458281, 0.07758059914 \pm i1.387412147\}. \quad (64)$$

Writing G^{-1} for r.h.s.(60), we now have

$$|G^{-1}(g_0)| \in \{0.285891853, 3.235626655\}, \quad (65)$$

and the smaller of these two values is the radius of convergence of $G(\xi)$.

Lastly, we translate the radius of convergence $\xi^* = 0.285891853$ of the series $G(\xi)$ given in (59) into a sufficient criterion of convergence of our formal perturbation series $D = D_C + \sum_{k=1}^{\infty} \beta^{4k} D^{(k)}$ and $H = H_A + \sum_{k=1}^{\infty} \beta^{4k} H^{(k)}$. Namely, if

$$\beta^2(\|D_C\| + \|H_A\|) < 0.285891853 \quad (66)$$

then our perturbative power series solution pair for D and H converges to a classical solution for the stationary Maxwell–Born–Infeld field equations.

This completes our convergence proof.

5 Purely electric or magnetic fields

When $j \equiv 0$ or $\rho \equiv 0$, then the perturbation theory for solving the stationary electromagnetic Maxwell–Born–Infeld equations simplifies considerably. In particular, when applying the perturbation theory directly to the electrostatic or magnetostatic field equations, better convergence estimates are obtained than by simply specializing the electromagnetic estimates to these cases.

5.1 The electrostatic case

When $j \equiv 0$ and H vanishes at spatial infinity we have $B \equiv 0 \equiv H$ and the stationary Maxwell–Born–Infeld equations reduce to the electrostatic Maxwell–Born(–Infeld) equations, comprising Coulomb’s law

$$\nabla \cdot D = 4\pi\rho, \quad (67)$$

and the stationary Faraday law

$$\nabla \times E = 0. \quad (68)$$

The two fields E and D are linked by the aether law of Born [Bor1933, Bor1969],

$$E = \frac{D}{\sqrt{1 + \beta^4 |D|^2}} \quad \text{Born's E – law.} \quad (69)$$

These equations are covered in [CaKi2010]. For the convenience of the reader we summarize the main results in the notation of the present paper.

Thus, inserting our perturbation theory Ansatz $D = D_C + D_\beta$, with $D_\beta = \beta^4 D^{(1)} + \beta^8 D^{(2)} + \dots$, into (69) and collecting powers of β , we find that $D^{(k)}$ for $k \in \mathbb{N}$ satisfies the pair of linear first-order PDE

$$\nabla \cdot D^{(k)} = 0 \quad (70)$$

$$\nabla \times D^{(k)} = \nabla \times V^{(k)}, \quad (71)$$

where $V^{(k)}$ is a polynomial in the $D^{(\ell)}$ with $\ell < k$, viz.

$$V^{(k)} = - \sum_{h=1}^k D^{(k-h)} \sum_{j=1}^h M_j \sum_{\substack{|\ell|_{2j} = \\ h-j}} \prod_{i=1}^j D^{(\ell_{2i-1})} \cdot D^{(\ell_{2i})}, \quad (72)$$

where $|\ell|_{2j}$ and M_j have their earlier assigned meaning. The pair of linear first-order PDE (70), (71) has a unique solution in $C_0^{1,\alpha}$ given by

$$D^{(k)} = \mathbf{P}V^{(k)}, \quad k \in \mathbb{N} \quad (73)$$

where $\mathbf{P}: C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ projects onto the solenoidal subspace of $C_0^{1,\alpha}$; see (30). The first member of this hierarchy reads explicitly

$$D^{(1)}(x) = \frac{1}{2}|D_C|^2 D_C + \nabla \int \frac{\frac{1}{4\pi} \nabla \cdot (\frac{1}{2}|D_C|^2 D_C)(y)}{|x-y|} d^3y, \quad (74)$$

so $\beta^4(74)$ gives the leading correction to the Coulomb term $D^{(0)} = D_C$.

In general, the perturbation series $D = \sum_{k=0}^{\infty} \beta^{4k} D^{(k)}$, with $D^{(0)} = D_C$ given by (10) and $D^{(k)}$ for $k \in \mathbb{N}$ recursively given by (73), is a *formal* series solution for the electrostatic field problem (67), (68) with prescribed ρ . Following essentially the same strategy, in [CaKi2010] the convergence in $C_0^{1,\alpha}$ of the formal power series is proved for

$$\beta^2 \|D_C\| < (2^{2/3} - 1)^{3/2} \approx 0.4501964645, \quad (75)$$

in which case the perturbation series yields a classical solution D .

5.2 The magnetostatic case

When $\rho \equiv 0$ and D vanishes at spatial infinity we have $D \equiv 0 \equiv E$ and the stationary Maxwell–Born–Infeld equations reduce to the magnetostatic Maxwell–Born(–Infeld) field equations, comprising the stationary Ampère law

$$\nabla \times H = 4\pi j \quad (76)$$

and the law of the absence of sources of the magnetic induction field B ,

$$\nabla \cdot B = 0. \quad (77)$$

The two fields B and H are linked by the aether law of Born [Bor1933, Bor1969],

$$B = \frac{H}{\sqrt{1 - \beta^4 |H|^2}} \quad \text{Born's B – law.} \quad (78)$$

While these equations are not directly covered in [CaKi2010], they can be treated almost verbatim to the treatment in [CaKi2010]. The minus sign under the square root is already covered in [CaKi2010], and otherwise only the roles of the $\nabla \cdot$ and $\nabla \times$ operators need to be exchanged.

Thus, inserting our perturbation theory Ansatz $H = H_A + H_\beta$, with $H_\beta = \beta^4 H^{(1)} + \beta^8 H^{(2)} + \dots$, into (78) and collecting powers of β , we find that $H^{(k)}$ for $k \in \mathbb{N}$ satisfies the pair of linear first-order PDE

$$\nabla \times H^{(k)} = 0, \quad (79)$$

$$\nabla \cdot H^{(k)} = \nabla \cdot U^{(k)}, \quad (80)$$

where $U^{(k)}$ is a polynomial in the $H^{(\ell)}$ with $\ell < k$, viz.

$$U^{(k)} = - \sum_{h=1}^k H^{(k-h)} \sum_{j=1}^h |M_j| \sum_{\substack{|\ell|_{2j}= \\ h-j}} \prod_{i=1}^j H^{(\ell_{2i-1})} \cdot H^{(\ell_{2i})}, \quad (81)$$

where again $|\ell|_{2j}$ and M_j have their earlier assigned meaning. The pair of linear first-order PDE (79), (80) has a unique solution in $C_0^{1,\alpha}$ given by

$$H^{(k)} = \mathbf{Q} U^{(k)}, \quad k \in \mathbb{N} \quad (82)$$

where $\mathbf{Q}: C_0^{1,\alpha} \rightarrow C_0^{1,\alpha}$ projects onto the curl-free subspace of $C_0^{1,\alpha}$; see (31). The first member of this hierarchy reads explicitly

$$H^{(1)}(x) = \frac{1}{2} |H_A|^2 H_A - \nabla \times \int \frac{\frac{1}{4\pi} \nabla \times (\frac{1}{2} |H_A|^2 H_A)(y)}{|x-y|} d^3 y, \quad (83)$$

so $\beta^4(83)$ gives the leading correction to the Ampère term $H^{(0)} = H_A$.

In general, the perturbation series $H = \sum_{k=0}^{\infty} \beta^{4k} H^{(k)}$, with $H^{(0)} = H_A$ given by (11) and $H^{(k)}$ for $k \in \mathbb{N}$ recursively given by (82), is a *formal* series solution for the magnetostatic field problem (76), (77) with prescribed j . Following nearly verbatim the strategy in [CaKi2010] the convergence in $C_0^{1,\alpha}$ of the formal power series can be proved for

$$\beta^2 \|H_A\| < (2^{2/3} - 1)^{3/2}, \quad (84)$$

in which case the perturbation series yields a classical solution H .

6 Summary and outlook

We have constructed a convergent perturbative power series which solves the stationary Maxwell–Born–Infeld equations with regular sources ρ and j in $C_0^{0,\alpha}$ as long as $\beta^2(\|D_C\| + \|H_A\|) < \xi^* = 0.28$ (approximately); here, $\|\cdot\|$ is the $C_0^{1,\alpha}$ norm, D_C the Coulomb field and H_A the Ampère field. In the absence of either D_C or H_A the corresponding perturbative power series for the remaining field even converges when $\beta^2\mathcal{N} < \xi_* \approx 0.45$, where \mathcal{N} is the $C_0^{1,\alpha}$ norm of H_A , resp. D_C ; cf. [CaKi2010].

While our solution method does not cover the conceptually important case of delta function sources for which Born conceived of this nonlinear field theory [Bor1933, Bor1937, BoIn1933], it nevertheless allows one to gain some relevant insight into a burning open question: *reliable bounds on the size of Born’s aether constant β* .

In a purely electrostatic calculation Born and Infeld [BoIn1933] computed $\beta = \beta_B \approx 1.236\alpha_S$ (in the units of [Kie2004a, Kie2004b], where $\alpha_S \approx 1/137.036$ is Sommerfeld’s fine-structure constant; so, $\beta_B \approx 0.00902$). Subsequently, trying to take the magnetic moment of the electron into account, Born and Schrödinger [BoSchr1935] came up with a purely magnetostatic (very rough) estimate of $\beta = \beta_{BS} \approx 4.74\beta_B$. Subsequently Rao [Rao1936] undertook a more detailed electromagnetic study, following Born’s suggestion to treat the electron not as a point but as a charged, rotating (one-dimensional) ring. However, all these plausible estimates are not based on an experimentally accessible process, involving as they do solely a single “stationary electron solution” in an otherwise empty space. Various attempts have been made since to compute effects on atomic spectra induced by the Born–Infeld nonlinearity to estimate the range of viable β values (see [CaKi2006] and references therein; see also [FrGa2010] for a recent study), but all these methods are based on various as-of-yet uncontrolled approximations and have produced conflicting results.

Our solution method now paves the way for a systematic study based on controlled approximations. For example, if we replace the Dirac delta function of each point charge by a $C^{0,1}$ function supported on a ball of radius equal to the Compton wavelength of the electron (which sets the unit of length in [Kie2004a, Kie2004b]), then $\|D_C\| \approx 3\alpha_S$, so our electrostatic series will converge for $\beta \lesssim 0.3874/\sqrt{\alpha_S} \approx 4.5348$, which range of β values safely includes the value proposed by Born. In particular, our perturbation series should converge sufficiently rapidly so that only a few terms need to be computed; see the appendix. Of course, to not compare apples with oranges, the resulting spectral data computed for the Schrödinger operators obtained with the help of these systematically computed $D = D_C + D_\beta$ fields for smeared

out charges should be compared with those computed with the Schrödinger operator with Coulomb potential for the same “smeared-out” charges, not point charges.

Similarly, we may also try to estimate the viable range of β values for an electron model with magnetic moment, although comparison with the purely magnetostatic calculation by Born–Schrödinger will at best be an academic exercise because such a model (presumably) can’t produce the Hydrogen spectrum in the $\beta \rightarrow 0$ limit. Instead one should base the assessment on the computation of a stationary electromagnetic solution with both ρ and j sources supported on a “smeared-out” ring, in the spirit of Rao’s study [Rao1936]. However, one possible caveat is that the β value may get uncomfortably close to the radius of convergence of the electromagnetic series, so that more terms may need to be calculated than is practically feasible.

We close the main body of this paper with an outlook on desired generalizations of the perturbative solution strategy pursued here for the stationary Maxwell–Born–Infeld field equations in three-dimensional Euclidean space. The generalization to the stationary model in higher-dimensions, which in recent years has become of interest to the high energy theory community (see, e.g., [Gib1998, LiYa2003]), should be quite straightforward (cp. the electrostatic version in [CaKi2010]). Less straightforward, but very desirable, is the generalization to the genuinely time-dependent setting; see [Ser1988, Ser2004, Bren2004, Spe2010] for recent rigorous results, [Boi1970] for an early pioneering study, and [BiBi1983] for a delightful general discussion. The main problem to be overcome to make the perturbative strategy work in the dynamical setting is the identification of a suitable Banach algebra. Lastly, and most importantly from the Born–Infeld perspective [Bor1933, BoIn1933], it is desirable to have the method which can handle delta function sources both in flat space (see [Bor1933, BoIn1933, Eck1986] and [Hop1994, Gib1998] for some explicit solutions with a single, respectively infinitely many point charges) and in curved spacetime (see [Hof1935a, Hof1935b, GSP1984, Dem1986, Bret2001, TaZa2010] for explicit solutions with a single point charge). The technique presented in this paper can serve to produce seed field-configurations for growing a solution for delta function sources in a suitable limiting process. Progress in this direction will be reported elsewhere.

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Appendix¹

Practically only a small number of terms of the perturbation series for D and H can be computed. Here we supply some details about the recursion coefficients R_k which are needed in the error estimates. Given the input value $R_0 = 1$, the author computed R_1, R_2, R_3 by hand from the recursion formula given in the main text, then used MAPLE to compute 14 terms using the iteration of the fixed point map given by the implicit definition of $G(\xi)$ in a direct, unpolished manner (after which MAPLE reached its capacity). The more sophisticatedly iterating Shalosh B. Ekhad knows 303 terms of the sequence $\{R_k\}_{k \in \mathbb{N} \cup \{0\}}$; here are the first 30 of them:

$$R_0 = 1 \tag{85}$$

$$R_1 = \frac{3}{2} \tag{86}$$

$$R_2 = \frac{65}{8} \tag{87}$$

$$R_3 = \frac{943}{16} \tag{88}$$

$$R_4 = \frac{62689}{128} \tag{89}$$

$$R_5 = \frac{1128197}{256} \tag{90}$$

$$R_6 = \frac{42790845}{1024} \tag{91}$$

$$R_7 = \frac{842157399}{2048} \tag{92}$$

$$R_8 = \frac{136312116961}{32768} \tag{93}$$

$$R_9 = \frac{2817640708457}{65536} \tag{94}$$

$$R_{10} = \frac{118490151386655}{262144} \tag{95}$$

$$R_{11} = \frac{2526390089218393}{524288} \tag{96}$$

$$R_{12} = \frac{217977129447815405}{4194304} \tag{97}$$

¹**Disclaimer:** Everything in this appendix could be made rigorous, but this here is only semi-rigorous. The whole process took Ekhad 200.542 seconds.

$$R_{13} = \frac{4748017126294329161}{8388608} \quad (98)$$

$$R_{14} = \frac{208584441836961908949}{33554432} \quad (99)$$

$$R_{15} = \frac{4614991020517094410279}{67108864} \quad (100)$$

$$R_{16} = \frac{1644116252728526666074977}{2147483648} \quad (101)$$

$$R_{17} = \frac{36812969231234813601419473}{4294967296} \quad (102)$$

$$R_{18} = \frac{1656740336870818323274233515}{17179869184} \quad (103)$$

$$R_{19} = \frac{37445969415289365495538129125}{34359738368} \quad (104)$$

$$R_{20} = \frac{3398982473269915594232889691503}{274877906944} \quad (105)$$

$$R_{21} = \frac{77410530113072758320538102052283}{549755813888} \quad (106)$$

$$R_{22} = \frac{3537571318060518004220386126288923}{2199023255552} \quad (107)$$

$$R_{23} = \frac{81073698522885193141789999978889377}{4398046511104} \quad (108)$$

$$R_{24} = \frac{14905122955618940253574385037312323437}{70368744177664} \quad (109)$$

$$R_{25} = \frac{343396823100629008332240991968973221221}{140737488355328} \quad (110)$$

$$R_{26} = \frac{15859792436328056179243840618808531803115}{562949953421312} \quad (111)$$

$$R_{27} = \frac{367032135637139188746851720898633881475805}{1125899906842624} \quad (112)$$

$$R_{28} = \frac{34043872901750574940303463222092806831775365}{9007199254740992} \quad (113)$$

$$R_{29} = \frac{790900804799227563339255535454710815651794705}{18014398509481984}. \quad (114)$$

For many practical purposes the above list should be fully sufficient. However,

should more terms be needed it is more efficient to switch to an asymptotic representation. Unfortunately, the recursion formula found in this paper is not practically useful for this purpose, for it quickly grows out of control because it involves *all* previous R_ℓ values ($\ell < k$) to compute the k -th one. And the fix point iteration produces the R_k without yielding control of their asymptotics. Fortunately, we know from general theorems that the associated sequence of integers $\{S_{k+1} := 2^{2k} R_k\}_{k \in \mathbb{N} \cup \{0\}}$ satisfies a linear recurrence of fixed finite order with coefficients which are fixed polynomials in $n = k + 1 \in \mathbb{N}$. The beginning of this integer sequence reads $\{1, 6, 130, 3772, 125378, \dots\}$. After a few minutes of interaction with Doron Zeilberger, Shalosh B. Ekhad found: given $\{S_1, \dots, S_9\}$, the ensuing S_n values are obtained as

$$S_{n+1} = \sum_{h=0}^8 P_h(n) S_{n-h}, \quad (115)$$

where each P_h is a polynomial of degree 9 in n , with integer coefficients. The coefficients of these polynomials are insanely large, with more than 100 digits, so that it would be pointless to list them here; they are available from Dr.Z.'s computer.

The asymptotic behavior of $\{S_{k+1}\}_{k \in \mathbb{N} \cup \{0\}}$ can now be extracted from the linear recurrence. The leading term in the asymptotics is determined by the degree-8 polynomial in the complex variable z formed by collecting the coefficients of highest power (i.e. n^9) of the polynomials $P_h(n)$ in (115), each multiplied by the associated h -th power of z , thus (symbolically)

$$\tilde{P}(z) = \sum_{h=0}^8 \frac{1}{9!} P_h^{(9)}(n) z^h, \quad (116)$$

where $P_h^{(9)}(n)$ is the 9-th derivative of $P_h(n)$, with n formally treated as real variable. As $n \rightarrow \infty$, the ratio S_{n+1}/S_n converges to the largest root of (116), which Ekhad easily finds as ≈ 48.9391511596 (truncated after 10 digits). Dividing by 4 yields

$$\lim_{k \rightarrow \infty} R_{k+1}/R_k = q \approx 12.23478778990. \quad (117)$$

As many digits of the ratio q are practically reached only for $k \gg 30$; however, empirically, R_{k+1}/R_k is monotonic increasing, and so

$$R_k < q^k. \quad (118)$$

This suffices to obtain the relevant error estimates for the truncation of the series.

Incidentally, by the ratio criterion, (117) yields the radius of convergence of $G(\xi)$ as $\xi^* = 1/\sqrt{q} \approx 0.285891853$, reproducing what we found earlier quasi-algebraically.

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